STABILITY ANALYSIS OF EQUILIBRIUM PATTERNS IN A TRANSPORTATION NETWORK

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Abstract

A general deterministic process models is described, for carrying out an equilibrium stability analysis, concerning both identification of attractors different from fixed-points and of bifurcations.
Main goal of this paper is to show that stability of equilibrium cannot be given for granted, in other words the equilibrium approach to demand assignment may fail to describe the state of the system relevant to analysis and design.

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1. INTRODUCTION

Assignment models, which simulate demand-supply interaction, are the basic tool to analyse and design transportation networks. Most existing models, and those commonly adopted in practical applications, follow an equilibrium approach, where it is assumed that mutually consistent flows and costs well describe the state of the system relevant for analysis and design (a recent review in Cantarella and Cascetta, 2001). More general models, derived from dynamic process theory, explicitly simulate the evolution over time of the system, and the convergence to different types of attractors (a general framework in Cantarella and Cascetta, 1995; see also Cascetta, Coppola, Adamo, 2000).

Dynamic process models include an explicit simulation of user cost and choice updating processes underlying system evolution over time. Several authors have contributed to this increasingly interesting field through applying and analysing deterministic process models (main references are Smith, 1984; Horowitz, 1984; Cantarella, 1993; Cantarella and Cascetta, 1995) derived from discrete-time Non-Linear Dynamic Systems Theory (see among many others Glendinning P., 1999). Some authors have also proposed stochastic process models (Cascetta, 1989; Davis, Nihan, 1993; Hazelton, 2002; Watling 1996, 1999, 2002).

In this paper, a general deterministic process models is described, such that the equilibrium pattern is a fixed-point state. A simple but effective approach, based on exponential smoothing, is followed to model both user cost and choice updating processes. Then, conditions assuring the stability of a fixed-point state, which means that it is an attractor, are analysed with respect to main parameters, such demand, behaviour dispersion, derivatives of link cost functions, etc. as well as habit and yesterday experience weight. Presented theoretical expectations are confirmed by results obtained from numerical simulations for small networks, reported in previous papers (Cantarella, Velonà, 2000, 2001 e 2002).

First, main notations and basic definitions are introduced together with basic data concerning small examples addressed through the paper (section 2). Then conditions about parameters to assure fixed-point stability (section 3) are discussed, and a bifurcation analysis is carried out for fixed-point attractors, to investigate the type of attractor that may occur when a fixed-point state loses its stability (section 4). Relationship with equilibrium uniqueness conditions has also been discussed. Attractor identification through Poincarè characteristic multipliers is also briefly addressed (section 5), The results in this paper indicate that the equilibrium approach, which does not allow for an explicit stability analysis, may fail to effectively described the state of the system relevant to analysis and design.

2. DYNAMIC PROCESS MODELS

In this section first main notations and basic definitions about discrete-time non-linear dynamic systems are introduced after a briefly review of the mathematical background.

2.1. Mathematical background

A discrete-time non-linear dynamic system, also called a deterministic process, can be described by a recursive equation:

$$\mathbf{x}_t = \varphi(\mathbf{x}_{t-1}; q)$$

where

$\mathbf{x}_t \in X$ are the state variables, which describe the state of the system at time t within the feasible state set or state space X, $\mathbf{x}_0$ being the starting state;

q are the control parameters of the system;

$\varphi(\bullet)$ is the transition function.

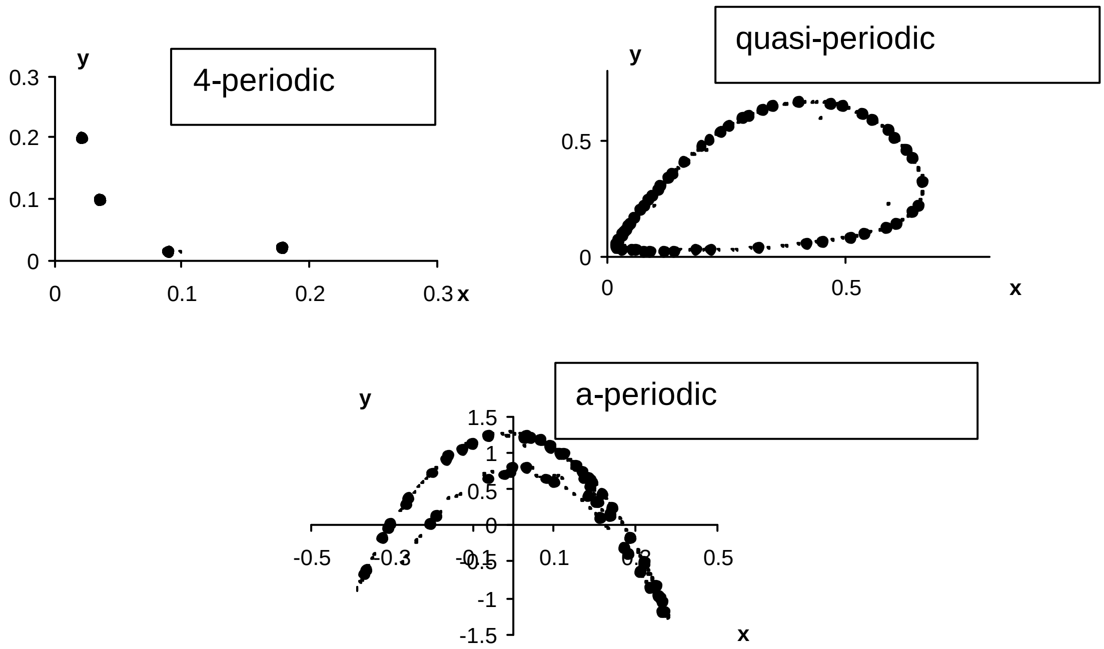
A dynamic system is called globally (or partially) *dissipative* if the state space (or a sub-set of it) tends to reduce to a null measure set, that is a set with a dimension smaller than the state space, called an *attractor*. More formally a subset A of the state space X is an attractor if:

- A has a dimension less than the state space $X \subseteq \Re^n$,
- starting within A the system state may not exit A,
- there exists a proper super-set B_A of A, called *basin*, such that starting within B_A the system state tends to A,
- A is minimal, that is it does not properly contain any sub-set with the above features.

Assuming that the transition function is continuous with its first partial derivatives, its Jacobian matrix $\mathbf{J}(\mathbf{x}_t) = \mathbf{Jac}[\varphi(\mathbf{x}_t)]$ can properly be defined. In this case a necessary condition (also sufficient for known dynamic non-linear systems, even though no formal proof exists to author's knowledge) to a system being dissipative over all the space state, is that the determinant of Jacobian matrix is less than one for any state, $|\mathbf{Jac}[\varphi(\mathbf{x})]| < 1,\ \forall\ \mathbf{x} \in X$ (see Thompson and Bishop, 1994, for a bi-dimensional system).

Four main types of attractors can be observed, according to taxonomy in the following table. Examples are given in figures below.

Type	# of states	Dimension	Geometry
fixed-point	1	= 0	One point
k-periodic	k	= 0	Several points
quasi-periodic	∞	> 0 integer	Torus
a-periodic	∞	> 0 non-integer	Fractal



A fixed-point attractor is made up by a single state $\mathbf{x}^* = \varphi(\mathbf{x}^*)$, and as such it may be considered a special case of k-periodic attractor: $\mathbf{x}_t = \varphi^k(\mathbf{x}_{t-k})$. Moreover, it should be noted that a fixed-point state may well not be an attractor, since it may occur that no basin exists for it.

Fixed-point, k-periodic and quasi-periodic attractors are non-chaotic, meaning that two evolutions starting close (enough) remain close; the system state evolves toward any of them through successive contraction over all directions. An a-periodic attractor is *chaotic* (for known dynamic non-linear systems, even though no formal proof exists to author's knowledge), that is however close two evolutions start they will greatly diverge after some time; the system state evolves toward it through successive contraction on some directions and stretching and folding on others, even though contracting as whole.

2.2. Dynamic process models for traffic assignment

Dynamic process models for traffic assignment refer to given period of the day (for instance morning rush hour in a working day) and studies the evolution form one day to next one of flows and costs, due to user updating of forecasted costs and choices. In this paper an exponential smoothing approach is followed to specify the transition function (a general framework in Cantarella and Cascetta, 1995).

The transportation supply is assumed modelled through a network, with m nodes, including an origin and a destination node for each considered traffic zone, and n arcs. Each day t, users are faced with path choice (as well as other choices such as mode, not addressed here for brevity's sake). At this aim users are assumed grouped into classes, each class i with the same origin–destination pair, sharing a common set K_i of available paths as well as any other behavioural characteristics. Paths are assumed numbered in such a way that a path $k \in K_i$ is univocally associated to a class i. let

$d_i \geq 0$ be the demand flow for user class i;

$\mathbf{y}_t \geq 0$ be the vector of path flows at day t, made up by a sub-vector $\mathbf{y}_{t,i}$ for each class i, with $\mathbf{1}^T \mathbf{y}_{t,i} = d_i$; the feasible path flow set being clearly compact (since bound and closed) and convex, as well as non-empty if at least one path connects each user class (sets K_i are non-empty);

$\mathbf{x}_t \geq 0$ be the vector of forecasted path cost at day t, made up by a sub-vector $\mathbf{x}_{t,i}$ for each class i; they are the result of user memory and learning;

$\mathbf{g}_t \geq 0$ be the vector of actual path cost at day t, made up by a sub-vector $\mathbf{g}_{t,i}$ for each class i.

Several approaches can be followed to specify cost updating process (a general framework in Cantarella and Cascetta, 1995). A simple but effective approach is based on exponential smoothing, which can lead to effective as well as easy to handle models:

$$\mathbf{x}_t = \beta\, \mathbf{g}_{t-1} + (1 - \beta)\, \mathbf{x}_{t-1}$$

where

$\beta \in (0,1]$ is the weight given to yesterday actual costs when forecasting costs today, $\beta = 0$ meaning no updating at all.

Path flows affects actually path costs due to the network structure:

$$\mathbf{f}_t = \Delta\, \mathbf{y}_t$$
$$\mathbf{c}_t = c(\mathbf{f}_t)$$
$$\mathbf{g}_t = \Delta^T \mathbf{c}_t$$

where

Δ is arc-path incidence matrix, made up by a sub-matrix Δ_i for each class i;

$\mathbf{f}_t \geq 0$ is the vector of arc flows at day t;

$c(\bullet)$ is the link cost flow (vector) function, assumed continuous with its partial first derivatives;

$\mathbf{c}_t \geq 0$ is the vector of actual arc cost at day t.

A exponential smoothing approach can be also followed to specify the choice updating process (a general framework in Cantarella and Cascetta, 1995):

$$\mathbf{y}_t = \alpha\, \mathbf{D}\, p(\mathbf{g}_t) + (1 - \alpha)\, \mathbf{y}_{t-1}$$

where

$\alpha \in (0,1]$ is the probability of reconsidering the previous day choice (but not necessarily change it), $\alpha = 0$ meaning no updating at all.

$\mathbf{D}$ is the (diagonal) flow demand matrix with as many row or columns as paths, and a block (m_i x m_i) $\mathbf{D}_{[i]} = d_i\, \mathbf{I}$ for each user class i;

$p(\bullet)$ is the path choice model, which expresses today path choice probabilities for user who have reconsider yesterday choice, for instance well-know Logit or Probit models.

By combining all the above equations a dynamic process model is obtained, which can be reformulated with respect to arc flows and forecasted costs:

$\mathbf{z}_t = \beta\, c(\mathbf{f}_t) + (1 - \beta)\, \mathbf{z}_{t-1}$ (1)

$\mathbf{f}_t = \alpha\, f(\mathbf{z}_t) + (1 - \alpha)\, \mathbf{f}_{t-1}$ (2)

where:

$\mathbf{z}_t \geq 0$ is the vector of forecasted arc costs at day t;

$f(\bullet) = \Delta\, \mathbf{D}\, p(\Delta^T \bullet)$ is the so-called network loading function between arc costs and flows (expressing the assignment to an un-congested network).

The system state is described by n + n variables, $\mathbf{z}_t$ and $\mathbf{f}_t$, the control parameters are those describing dynamic evolution, α and β, together with those within the cost-flow function, such as arc capacity and the like, and the network loading function, such as demand level, path choice parameters, etc.

The feasible arc flow set shows the same features of the feasible path flow set. Due to continuity of cost-flow function the feasible arc or path cost sets may be considered non-empty and compact, usually their convex closure is considered to get convexity. Generally, costs are considered non-negative.

Assuming that the cost-flow function, $c(\bullet)$, and the path choice model, $p(\bullet)$, are continuous with their first partial derivatives, their Jacobian matrices can properly be defined: $\mathbf{J}_c = \mathbf{Jac}[c(\bullet)]$ and $\mathbf{J}_p = \mathbf{Jac}[p(\bullet)]$. Then, the (2n×2n) Jacobian matrix of the transition function, $\mathbf{J}(\mathbf{z}_t, \mathbf{f}_t,)$ can properly be defined as:

$$\mathbf{J}(\mathbf{z}_t, \mathbf{f}_t,) = \begin{array}{|c|c|} \hline (1-\beta)\mathbf{I} & \beta\, \mathbf{J}_c \\ \hline \alpha\,(1-\beta)\, \mathbf{J}_f & (1-\alpha)\,\mathbf{I} + \alpha\beta\, \mathbf{J}_f\, \mathbf{J}_c \\ \hline \end{array} =$$

$$\begin{array}{|c|c|} \hline \mathbf{I} & \mathbf{0} \\ \hline \alpha\, \mathbf{J}_f & (1-\alpha)\mathbf{I} \\ \hline \end{array} \quad \text{x} \quad \begin{array}{|c|c|} \hline (1-\beta)\mathbf{I} & \beta\, \mathbf{J}_c \\ \hline \mathbf{0} & \mathbf{I} \\ \hline \end{array}$$

where

$\mathbf{J}_f = \mathrm{D}\mathbf{D}\, \mathbf{J}_p\, \mathrm{D}^T$ is the Jacobian matrix of the network loading function.

It is worth noting that if the network is not congested, $\mathbf{J}_c = \mathbf{0}$, or the users' choice behavior is completely random and thus does not depend on costs, $\mathbf{J}_p = \mathbf{0}$, the Jacobian matrix $\mathbf{J}$ at any point is triangular. In this case the eigenvalues of $\mathbf{J}$ are given by entries on main diagonal, $(1 - \alpha_k)$ or $(1 - \beta_k)$, and, having assumed $\alpha_k \in\,]0,1]$ and $\beta_k \in\,]0,1]$ the fixed-point is always stable.

The determinant of $\mathbf{J}(\mathbf{z}_t, \mathbf{f}_t,)$ is given by $(1 - \alpha)^n (1 - \beta)^n \in [0,1)$ since $\alpha \in (0,1]$ and $\beta \in (0,1]$. Hence the above system (1-2) is dissipative over all the space state, that is, as said above, it will converge to an attractor whichever the starting state. This consideration rules out the conditions $\alpha = 0$ or $\beta = 0$.

The fixed-point states, $(\mathbf{z}^*, \mathbf{f}^*)$ of the system (1-2), not necessarily attractors, are given by:

$\mathbf{z}^* = \beta\, c(\mathbf{f}^*) + (1 - \beta)\, \mathbf{z}^*$

$\mathbf{f}^* = \alpha\, f(\mathbf{z}^*) + (1 - \alpha)\, \mathbf{f}^*$

or

$\mathbf{z}^* = c(\mathbf{f}^*)$ and $\mathbf{f}^* = f(\mathbf{z}^*)$

It turns out to be equivalent to the well-known Stochastic User Equilibrium (SUE): $\mathbf{f}^* = f(c(\mathbf{f}^*))$. It should be noted that the dynamic control parameters, α and β, do not affect fixed-point states nor their existence and uniqueness.

2.3. Applications to simple networks

In this paper two simple applications referring to a two-link network or a three-link network are considered

A Logit path choice model is used in both cases:

$$p_j(\mathbf{g}) = \exp(-\theta g_i) / \sum_j \exp(-\theta g_j)$$

where

$\theta = \pi / (6^{1/2}\sigma) \cong 1.282/\sigma > 0$ and σ is the standard deviation of random residual, assumed independent of costs; so far the greater it is the less stochastic the user choice behaviour is.

Since the above choice probabilities actually depend on difference between path costs, rather than their values, one or two arc costs are enough to describes the cost pattern for a two- or a three-link network respectively. Moreover, due to demand conservation constraint one or two arc flows respectively, are enough to describes the flow pattern. Thus, the corresponding Jacobian matrix has two null eigenvalues.

The Logit scale parameter, θ, acts as a control parameter. In addition a demand level parameter, $\delta \geq 0$, will be referred, $\delta = 1$ meaning saturation. Details about cost-flow functions are in (Cantarella e Velonà, 2000 and 2002).

3. STABILITY ANALYSIS OF FIXED-POINT STATES

Even though existence and uniqueness of a fixed-point state can be assured (see Cantarella, 1997 for a discussion about sufficient conditions), if it is not (locally) stable the system may still evolve towards a different attractor, unless the starting state is the fixed-point (or equilibrium state) itself and no fluctuations, however small, occur. Thus, in this section, sufficient conditions concerning (local) stability, obtained from non linear dynamic system theory, will be presented.

3.1. Mathematical background

A fixed-point is (*asymptotically-locally*) *stable* if from any (sufficiently close) starting state the system state tends to the fixed-point as t tends to infinity (Wiggins, 1990).

Assuming that the transition function is continuous with its first partial derivatives, its Jacobian matrix $\mathbf{J}(\mathbf{x}_t) = \mathbf{Jac}[\varphi(\mathbf{x}_t)]$ can properly be defined. In this case, given a fixed-point, **x***, and the Jacobian matrix, $\mathbf{J}^* = \mathbf{Jac}[\varphi(\mathbf{x}^*)]$, the fixed-point, **x***, is stable, if all the eigenvalues of the Jacobian matrix **J*** have a modulus strictly less than one (see among others Wiggins, 1990):

$$|\lambda_j| < 1 \quad \forall j \quad \text{or} \quad \max_j |\lambda_j| < 1$$

where
λ_j is one of the eigenvalues of the Jacobian matrix **J***.

Hence, the stability region for eigenvalues λ_j is the interior of the unitary circle on the (Argand) complex plane:

$$(\lambda_{R,j})^2 + (\lambda_{Im,j})^2 < 1$$

where
$\lambda_{R,j}$ is the real part of the eigenvalue λ_j;
$\lambda\gamma_{Im,k}$ is the imaginary part of the eigenvalue λ.

if some eigenvalues have a modulus equal to one, $\max_j |\lambda_j| = 1$, stability should be studied through a second order analysis. This condition holds when a bifurcation occurs (see next section), clearly it is not *structurally stable*, say it vanishes with small changes of control parameters.

Generally, fixed-point stability should be studied by explicitly computing the Jacobian matrix at the fixed-point, and then its eigenvalues, since no general expression exists relating the eigenvalues of a matrix to its entries.

3.2. Stability analysis of equilibrium pattern

The above introduced conditions about fixed-point stability can be applied to the equivalent Stochastic User Equilibrium (SUE), considering the exponential smoothing dynamic process model described in sub-section 2.2.

In order to separate the effects of the dynamic control parameters, α and β, from those of the other control parameters, let us consider the ($n \times n$) matrix: $\mathbf{G} = \mathbf{J}_f \mathbf{J}_c = = \Delta\mathbf{D}\, \mathbf{J}_p \Delta^T \mathbf{J}_c$ within the Jacobian matrix $\mathbf{J}$; it is clearly independent of the dynamic control parameters, α and β, whilst it depends on all the other control parameters, through the cost-flow function and the path choice model. It can be shown (Cantarella and Cascetta, 1995) that for each eigenvalue γ_k of matrix $\mathbf{G}$ two eigenvalues of matrix $\mathbf{J}$ can be defined $\lambda'_k = \lambda_k$ and $\lambda''_k = \lambda_{m+k}$:

$$\lambda'_k = [(1 - \alpha) + (1 - \beta) + \alpha \beta \gamma_k - (\chi_k)^{1/2}] / 2$$
$$\lambda''_k = [(1 - \alpha) + (1 - \beta) + \alpha \beta \gamma_k + (\chi_k)^{1/2}] / 2$$

where

$$\chi_k = [(1 - \alpha) + (1 - \beta) + \alpha \beta \gamma_k]^2 - 4(1 - \alpha)(1 - \beta)$$

With reference to the Jacobian matrix $\mathbf{J}^*$ at the fixed-point ($\mathbf{z}^*$, $\mathbf{f}^*$), from the above equations the stability region for eigenvalues γ_k is the interior of an ellipse on the (Argand) complex plane:

$$((\gamma_{R,k} - 1) + e_R)^2 / (e_R)^2 + (\gamma_{Im,k})^2 / (e_{Im})^2 < 1$$

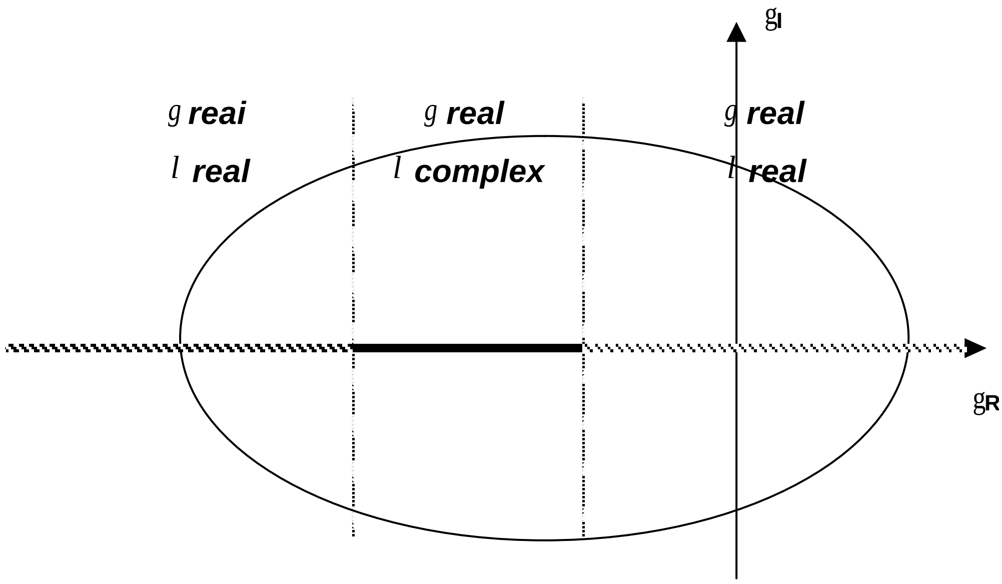
where

$\gamma_{R,k}$ is the real part of the eigenvalue γ_k;
$\gamma_{Im,k}$ is the imaginary part of the eigenvalue γ_k;
$e_R = [1 + (1 - \alpha)(1 - \beta)] / (\alpha\beta) \geq 1$ is the real semi-axis:
$e_{Im} = [1 - (1 - \alpha)(1 - \beta)] / (\alpha\beta) \geq 1$ is the imaginary semi-axis.

It turns out that two conjugate complex pairs of eigenvalues of $\mathbf{J}$ correspond to a conjugate complex pair of eigenvalues of $\mathbf{G}$, whilst two real eigenvalues of $\mathbf{J}$ or a conjugate complex pair of eigenvalues may correspond to a real eigenvalue of $\mathbf{G}$, as shown in figure below.

The ellipse depends on parameters α and β; the smaller parameters α and β are the greater the area within the ellipse is. It is worth noting that the effect of parameters α and β is symmetric, say they can be exchanged without affecting results.

In any case, the ellipse is symmetrical with respect to real axis, and cut it at $\gamma_{R,k} = 1$ and $\gamma_{R,k} = 1 - 2e_R \leq 0$. Hence, if any eigenvalue of $\mathbf{G}$ has real part greater than one, $\exists k\ \gamma_{R,k} > 1$, the fixed-point will be unstable whichever the values of parameters α and β. On the other hand if all the eigenvalues of $\mathbf{G}$ have real part less than one, $\gamma_{R,k} > 1\ \ \forall k$, there always exist small enough values of parameters α and β to assure stability.



3.3. Applications to simple networks

Some small examples of the application of the analysis carried out in sub-section 3.2 are discussed below. It is worth noting that the same qualitative behaviour would be observed in a real-size network, since stability is lost with respect to one real eigenvalues (or a conjugate complex pair) at time.

Results for a two-link network are discussed below, with α = 0.25, β = 0.90, δ = 0.40, θ = 0.22, 0.28, 0.40, the starting state resulting irrelevant. Figure 1 below reports the two non-null eigenvalues λ of **J**, the evolution over time of the system, and the attractor over the state space plan, given by one arc flow and one arc cost (as highlighted in sub-section 2.3). Both the eigenvalues λ are real, and depending on the value of parameter θ three cases occur:
i) the fixed-point is stable, both the eigenvalues are within the unitary circle;
ii) the fixed-point is non-stable, only one eigenvalue is inside the unitary circle, the system evolves towards a 2-periodic attractor;
iii) the fixed-point is non-stable, only one eigenvalue is inside the unitary circle, the system evolves towards an a-periodic attractor.

Results for a three-link network are discussed below, with α = 0.98, β = 0.98, δ = 0.4, θ = 0.010, 0.015, 0.018, The starting state resulting irrelevant. Figure below reports the two non-null eigenvalues γ of **G**, the evolution over time of the system, and the attractor over the flow space plan. The two eigenvalues γ occur in a complex conjugate pair, depending on the value of parameter θ three cases occur.
i) the fixed-point is stable, the two eigenvalues are within the stability ellipse;
ii) the fixed-point is non-stable, the two eigenvalues are outside the stability ellipse, the system evolves towards a quasi-periodic attractor;
iii) the fixed-point is non-stable, the two eigenvalues are outside the stability ellipse, the system evolves towards a 2-periodic attractor.

Similar results can be obtained by changing any other control parameter.

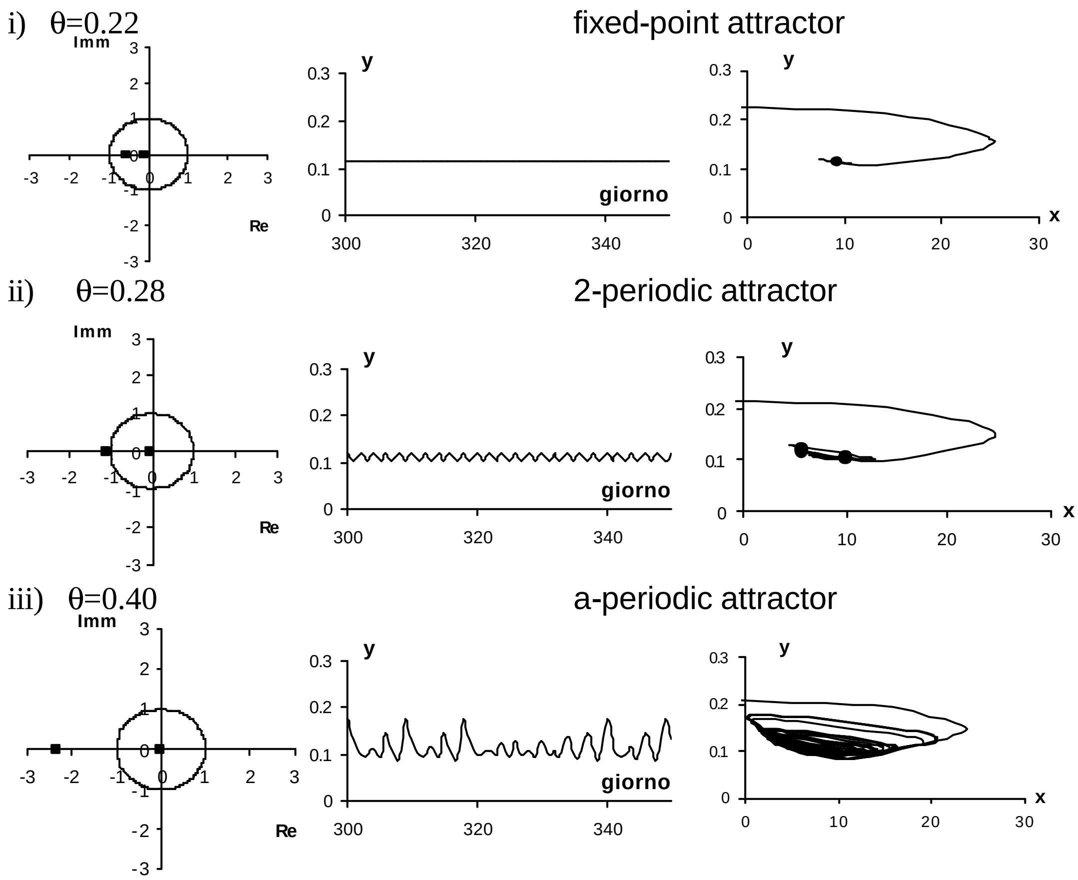


Figure 1: examples for a two-link network

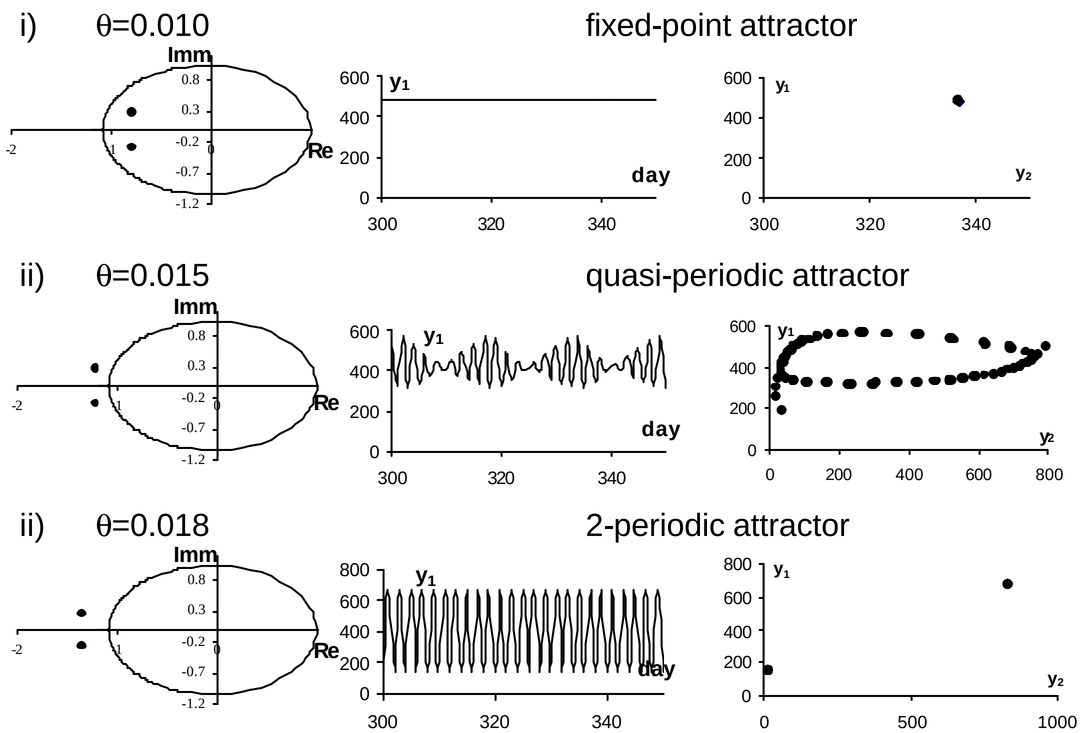


Figure 2: examples for a three-link network

4. BIFURCATIONS ANALYSIS OF FIXED-POINT STATES

A *bifurcation* occurs when small changes of control parameters determines relevant qualitative variations of the type of system evolution over time. For instance, as shown in the previous section, a stable fixed-point may become non-stable and a different type of attractor occurs. This section discussed a full bifurcation analysis (for fixed-point attractors only) formally supporting results of the examples reported in the previous section, where (sufficient) conditions for (local) stability of fixed-point states have been discussed.

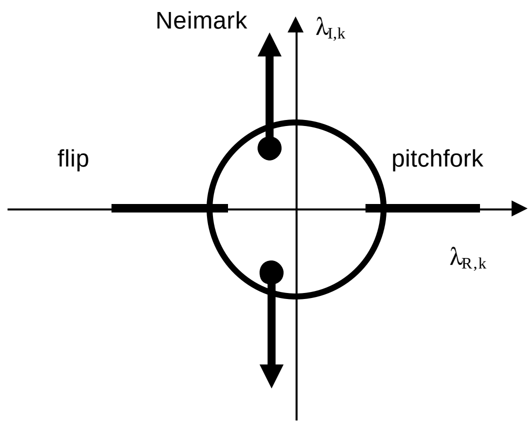
4.1. Mathematical background

Following a change in one (or more) control parameter a stable (in the sense defined in sub-section 3.1) fixed-point state may become non-stable if an eigenvalue λ_k of **J** gets a modulus greater than one, $|\lambda_k| > 1$; this condition can be reached through three different ways (see for instance Wiggins, 1990).

- One negative real eigenvalue λ_k of **J** become less than minus one: $\lambda_k < -1$; as the eigenvalue become smaller and smaller first 2-periodic, 4-periodic, 8-periodic, … attractors can be observed then an a-periodic one occurs, *flip bifurcation* (the sequence is described by the Feigenbaum cascade, see Thompson and Stewart,1986).

- The modulus of one complex conjugate pairs of eigenvalues λ_k, λ_{k+1} of **J** become greater than one: $|\lambda_k| > 1$ and $|\lambda_{k+1}| > 1$; a quasi-periodic attractor can be observed, *Neimark bifurcation*.

- One positive real eigenvalue λ_k of **J** become greater than one: $\lambda_k > 1$; in this case, for dissipative systems, two different fixed-point attractors occur, *pitchfork bifurcation* (for non-dissipative systems other bifurcations can be observed, see Thompson and Stewart 1986).

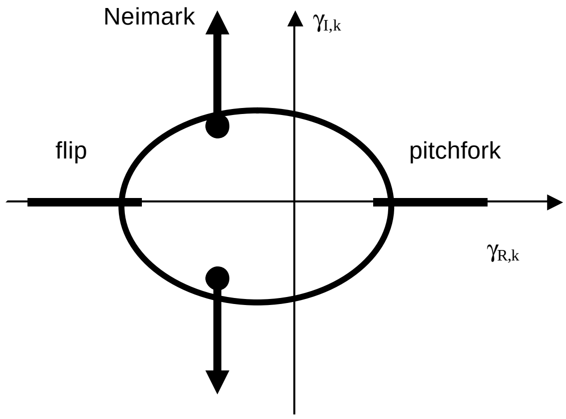


Clearly as the modulus of an eigenvalue becomes greater and greater than one, the moduli of other eigenvalues may be become greater than one. In this case the above described bifurcation may mixed together.

4.2. Bifurcation analysis of equilibrium pattern

The three types of bifurcations discussed in the previous sub-section can be observed with reference to the fixed-point state of a dynamic process model for traffic assignment. The analysis can be carried out with reference to the eigenvalues γ_k of matrix **G** with respect to the elliptical stability region, introduced in sub-section 3.2.

- *Flip bifurcation*: 2-periodic, 4-periodic… attractors then an a-periodic one, when $\gamma_k < 1 - 2e_R$ for a real eigenvalue γ_k;
- *Neimark bifurcation*: quasi-periodic attractor, when $|\gamma_k| > 1$ and $|\gamma_{k+1}| > 1$; for a complex conjugate pairs of eigenvalues γ_k, γ_{k+1};
- *Pitchfork bifurcation*: two new fixed-point attractors, when $\gamma_k > 1$ for a real eigenvalue γ_k.



Under mild assumptions (see Cantarella, 1997) the network loading function, $f(\mathbf{c})$, has a symmetric semi-definite Jacobian matrix, $\mathbf{J}_f$, in this case conditions on the Jacobian matrix, $\mathbf{J}_c$, of the cost-flow function, $c(\mathbf{f})$, may rule out some of the above presented bifurcations.

- If the Jacobian matrix $\mathbf{J}_c$ is positive definite (but not necessarily symmetric) the eigenvalues of matrix **G** can be proved with non-positive real parts, thus a pitchfork bifurcation cannot occur, ruling out the case of multiple fixed-point states; this result is consistent with sufficient uniqueness conditions about traffic equilibrium. It should be noted that it is a too strong condition which could be relaxed to eigenvalues with real parts less than one.
- If the Jacobian matrix $\mathbf{J}_c$ is also symmetric the eigenvalues of matrix **G** can be proved real, thus a Neimark bifurcation cannot occur, and that stability condition reduce to $\gamma_k > 1 - 2e_R$.

It should be also noted that in a two-link network with only two eigenvalues different from zero, since their product is equal to the determinant of matrix **G** that is less than one (the system being dissipative) a Neimark bifurcation cannot be observed.

4.3. Applications to simple networks

Some examples of the bifurcations presented in sub-section 4.2 are discussed below, with respect to the Logit parameter θ. Similar results can be obtained with respect to other parameters, and are not reported for brevity's sake. Again, the same qualitative behaviour would be observed in a real-size network, since stability is lost with respect to one real eigenvalues (or a conjugate complex pair) at time.

Results for a two-link network are presented below, with $\alpha = 0.25$, $\beta = 0.90$, $\delta = 0.40$, the starting state resulting irrelevant. Figure 3 below reports a flip bifurcation with respect to the Logit parameter. Results are consistent with those in figure 1.

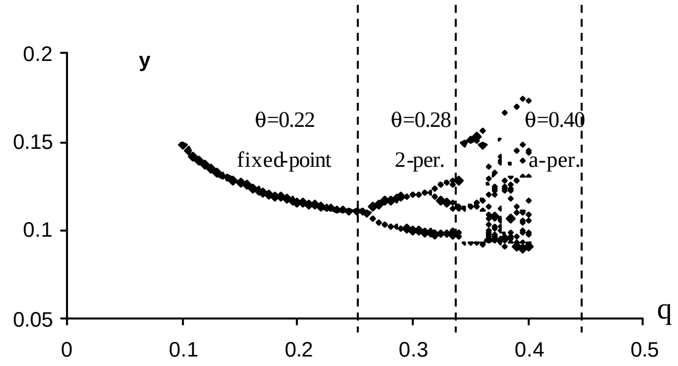
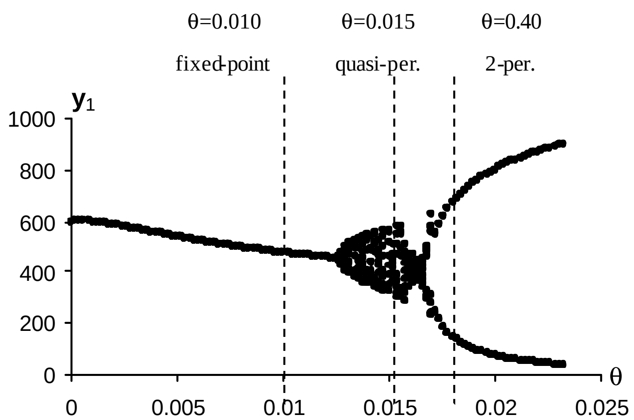


Figure 3: examples for a two-link network

Results for a three-link network are presented below, with $\alpha = 0.98$, $\beta = 0.98$, $\delta = 0.40$, the starting state resulting irrelevant. Figure 4 below reports a Neimark bifurcation with respect to the Logit parameter. Results are consistent with those in figure 2.



5. ATTRACTOR IDENTIFICATION

This section presents a method to identify attractors a part from the pictorial analysis in the previous sections.

5.1. Mathematical background

Given the starting state $\mathbf{x}_0$ (and control parameters θ) the attractor can be identified by the contracting or expanding factor for each direction, called *Lyapunov's* (or *Poincarè's* or *Floquet's*) multiplier, μ_j. For a transition function, $\varphi(\bullet)$, continuous with its partial first derivatives, they are given by:

$$\mu_j = \lim_{t\to\infty}(|\lambda^t_j|)^{1/t}$$

where

λ^t_j is the j-th eigenvalue of matrix $\mathbf{Jac}[\varphi^t(\mathbf{x}_0)]$ (which can be computed through the chain rule).

Conditions on multipliers define the type of attractors as in the table below, where multipliers are sorted in a descent order, $0 \le \mu_n \le \ldots \le \mu_k \le \ldots \le \mu_1$.

Type	Condition on multipliers
fixed-point	$0 \le \mu_n \le \ldots \le \mu_1 < 1$
k-periodic	$0 \le \mu_n \le \ldots \le \mu_1 < 1$
quasi-periodic	$0 \le \mu_n \le \ldots \le \mu_{k+1} < \mu_k = \ldots = \mu_1 = 1$ (the number of $\mu_k = 1$ gives the dimension of the torus)
a-periodic	$0 \le \mu_n \le \ldots \le \mu_1$ $\mu_1\, \mu_2 \ldots\ldots\, \mu_n < 1$

It should be noted that a fixed-point attractor cannot be distinguished from a k-periodic one, consistently with comments in sub-section 2.1.

A fixed-point or k-periodic or quasi-periodic attractor is called *hyperbolic* if it has dimension equal to the number of unitary multipliers. Non-hyperbolic attractors cannot be studied through multipliers, on other hand they are not *structurally stable*, say they vanish with small changes of control parameters.

It should be noted that condition $\mu_1\, \mu_2 \ldots\ldots\, \mu_n < 1$ holds for all type of attractors, even though not explicitly mentioned for the first three types, since clearly implied by the reported conditions. As said in sub-section 2.1 a condition to a system being dissipative is that the determinant of $\mathbf{Jac}[\varphi(\mathbf{x})]$ is less than one for any state, $|\mathbf{Jac}[\varphi(\mathbf{x})]| < 1,\ \forall\ \mathbf{x} \in X$. In this case, it clearly results $|\mathbf{Jac}[\varphi^t(\mathbf{x}_0)]| < 1, \forall \mathbf{x}_0 \in X$, thus confirming condition $\mu_1\, \mu_2 \ldots\ldots\, \mu_n < 1$.

5.2. Attractors identification for traffic assignment

Attractors of the dynamic process model (1-2) can be identify through the Lyapunov's multipliers with reference to the expression of the Jacobian matrix in sub-section 2.2. some examples are given in the next sub-section.

5.3. Applications to simple networks

For a two-link network, two non null eigenvalues should be considered, hence two multipliers should be analysed. Some examples are given in table below. Results are consistent with those in previous sub-sections.

θ	μ_1	μ_2	# of states
0.22	**0.000000**	**0.000001**	**1**
0.24	0.795186	0.000000	1
0.26	0.000000	0.985271	2
0.28	**0.000000**	**0.752689**	**2**
0.32	0.000000	0.000000	3
0.35	0.000000	0.000000	3
0.36	0.000000	0.772423	6
0.37	1.229642	0.000000	a-periodic
0.40	**0.000000**	**1.124728**	**a-periodic**
0.60	1.085361	0.000000	a-periodic
0.7	1.167977	0.000000	a-periodic
0.9	0.000000	0.880651	6
1	0.836746	0.000000	6

For a three-link network, our non null eigenvalues should be considered, hence four multipliers should be analysed, as shown below.

θ	μ_1	μ_1	μ_3	μ_4	# of states
0.007	0.774356	0.772992	0.772992	0.774356	1
0.010	0.886771	0.000001	0.901406	0.901406	1
0.012	0.955696	0.00000	0.972199	0.972199	1
0.013	0.000000	0.977098	1.000000	1.000000	quasi-periodic
0.014	0.000000	0.965666	1.000000	1.000000	quasi-periodic
0.015	1.000000	1.000000	0.000001	0.961409	quasi-periodic
0.018	0.863079	0.861943	0.000001	0.862810	2

6. CONCLUSIONS AND RESEARCH PERSPECTIVES

In this paper, the stability of the equilibrium flow (and cost) pattern coming out from traffic assignment has been analysed through a dynamic process model, based on exponential smoothing approach to cost and choice updating, by applying tools from discrete-time Non-linear Dynamic System theory.

It has been shown that stability of equilibrium cannot be given for granted, in other words the equilibrium approach to demand assignment may fail to describe the state of the system relevant to analysis and design.

Theoretical results have been discussed and illustrated through examples for simple networks. Results of applications for real-size networks, not reported here, numerically confirm results presented in this paper, but a more formally analysis seems useful as well as efficient algorithms for large scale applications and calibration against real data.

Some theoretical issues also seem worth of further research work, namely the analysis of the case of multiple fixed-point states, some of them stable other unstable, through results from catastrophe theory (as introduced by Thom, 1974), the length of transients before convergence, and the computation of a fractal measure for a-periodic attractors.

More generally, global stability, regarding the effect of the starting conditions, surely deserves an in-depth investigation, even though numerical results seem to support the conjecture that a locally stable fixed-point state is also globally stable.

Finally, more general deterministic process models deserve further analysis. Relationship with stochastic process models, in particular with the evolution over time of their expected values, is an other relevant field, which has already attract the interest of several researchers.

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STABILITY ANALYSIS OF EQUILIBRIUM PATTERNS IN A TRANSPORTATION NETWORK

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Introduction

Traffic assignment models

- Equilibrium
  - *no variations of demand and/or supply over time*

- (day-to-day) Dynamic process
  - *explicit modelling of cost and choice updating processes*

Dynamic process models

Mathematical background

discrete-time non-linear dynamic system

$$\mathbf{x}_t = \varphi(\mathbf{x}_{t-1}; q)$$

$\mathbf{x}_t \in X$ state variables
q control parameters of the system
$\varphi(\bullet)$ transition function

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Dynamic process models

Mathematical background

dissipative system : $|\mathbf{Jac}[\varphi(\mathbf{x})]| < 1, \forall\, \mathbf{x} \in X$

the state tends to an attractor *A*

- *A* has a dimension smaller than the state space
- the system state may not exit *A*
- there exists a basin B_A of *A*
- *A* is minimal

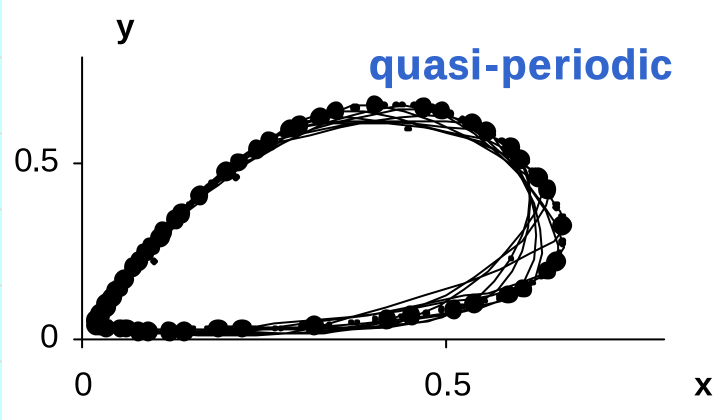
Type	# of states	Dimension	Geometry
fixed-point	1	= 0	One point
k-periodic	k	= 0	Several points
quasi-periodic	∞	> 0 integer	Torus
a-periodic	∞	> 0 non-integer	Fractal

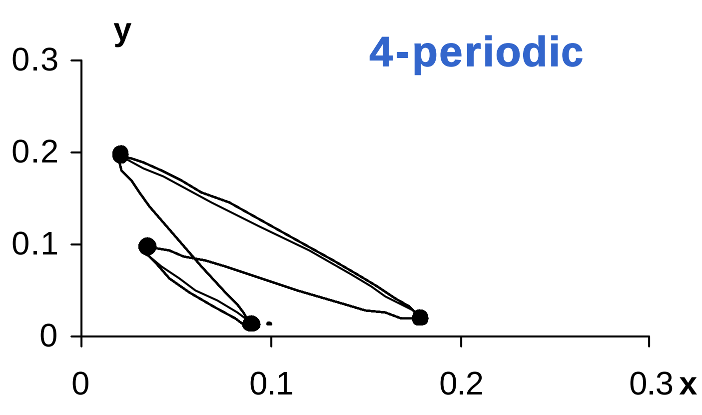
Dynamic process models

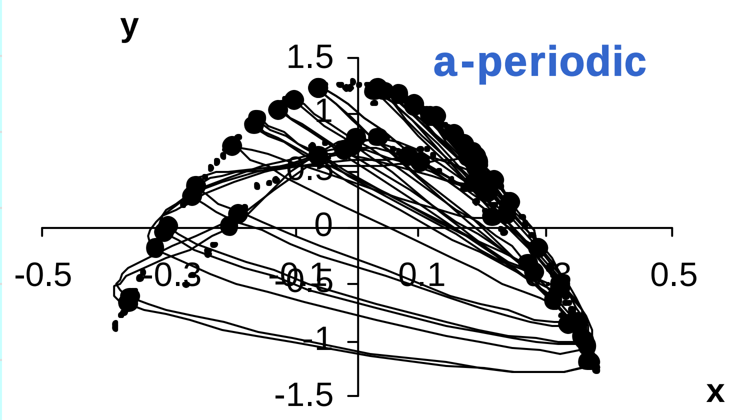
Mathematical background

fixed-point

$$\mathbf{x^*} = \varphi(\mathbf{x^*})$$







Static models

Traffic assignment

Cost flow function

$$\mathbf{c} = c(\mathbf{f})$$

$\mathbf{f} \geq 0$ arc flows
$\mathbf{c} \geq 0$ actual arc costs

$c(\bullet)$ arc (actual) cost flow (vector) function

Static models

Traffic assignment

Network loading function

$$\mathbf{f} = f(\mathbf{z}) = \Delta\, \mathbf{D}\, p(\Delta^T \mathbf{c})$$

c ≥ 0 arc costs
Δ arc-path incidence matrix
p($\bullet$) path choice model
D (diagonal) flow demand matrix
f ≥ 0 arc flows

$f(\bullet) = \Delta\, \mathbf{D}\, p(\Delta^T \bullet)$ network loading function

Dynamic process models

Traffic assignment

Cost updating process

$$\mathbf{z}_t = \beta\, c(\mathbf{f}_{t-1}) + (1 - \beta)\, \mathbf{z}_{t-1}$$

$\mathbf{z}_{t-1} \geq 0$ forecasted arc costs at day t-1
$\mathbf{f}_{t-1} \geq 0$ arc flows at day t-1
$\mathbf{z}_t \geq 0$ forecasted arc costs at day t

$c(\bullet)$ arc (actual) cost flow (vector) function

$\beta \in (0,1]$ weight given to yesterday actual costs

Dynamic process models

Traffic assignment

Choice updating process

$$\mathbf{f}_t = \alpha\, f(\mathbf{z}_t) + (1 - \alpha)\, \mathbf{f}_{t-1}$$

$\mathbf{f}_{t-1} \geq 0$ arc flows at day t-1
$\mathbf{z}_t \geq 0$ forecasted arc costs at day t
$\mathbf{f}_t \geq 0$ arc flows at day t

$f(\bullet)$ network loading function

$\alpha \in (0,1]$ prob. of reconsidering the previous day choice

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Dynamic process models

Traffic assignment

$$\mathbf{J}(\mathbf{z}_t, \mathbf{f}_{t,}) = \begin{array}{|c|c|} \hline (1-\beta)\mathbf{I} & \beta\,\mathbf{J}_c \\ \hline \alpha\,(1-\beta)\,\mathbf{J}_f & (1-\alpha)\,\mathbf{I} + \alpha\beta\,\mathbf{J}_f\,\mathbf{J}_c \\ \hline \end{array}$$

$\mathbf{J}_c = \mathbf{Jac}[c(\bullet)]$
$\mathbf{J}_p = \mathbf{Jac}[p(\bullet)]$
$\mathbf{J}_f = \mathbf{Jac}[f(\bullet)] =$
$= \Delta\mathbf{D}\ \mathbf{J}_p\ \Delta^T$ Jacobian matrix of network loading function

$|\mathbf{J}(\mathbf{z}_t, \mathbf{f}_{t,})| = (1-\alpha)^n (1-\beta)^n \in [0,1)$ DISSIPATIVE

Dynamic process models

Traffic assignment

fixed-point states

$$\mathbf{z}^* = \beta\, c(\mathbf{f}^*) + (1 - \beta)\, \mathbf{z}^*$$
$$\mathbf{f}^* = \alpha\, f(\mathbf{z}^*) + (1 - \alpha)\, \mathbf{f}^*$$

or

$$\mathbf{z}^* = c(\mathbf{f}^*)$$
$$\mathbf{f}^* = f(\mathbf{z}^*)$$

<u>equivalent</u> to SUE: $\mathbf{f}^* = f(c(\mathbf{f}^*))$

Stability analisys

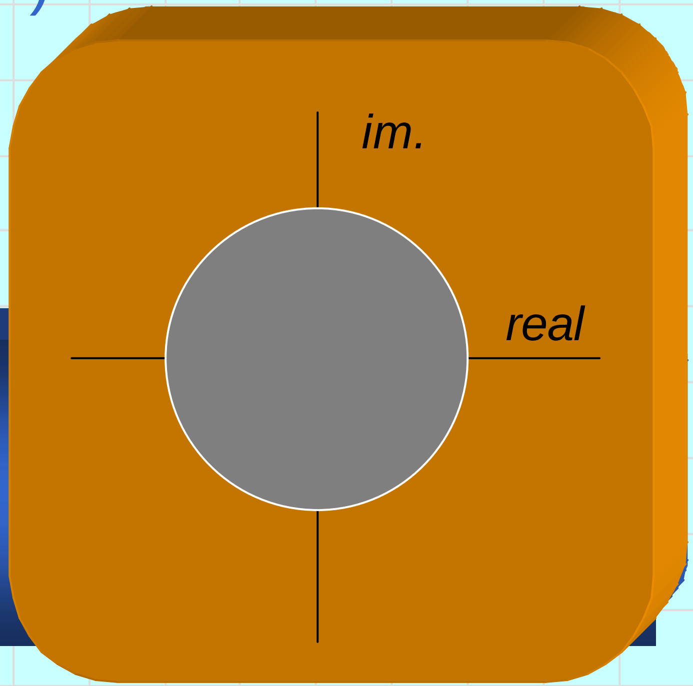
Mathematical background

$\mathbf{Jac}[\varphi(\mathbf{x})] = \partial\,(\mathbf{x}_t)\ /\ \partial\,(\mathbf{x}_{t-1})$

λ_j one of the eigenvalues of $\mathbf{J}(\mathbf{x}^*)$

- real
- complex coniugate pair

$$| \lambda_j | < 1 \qquad \forall j$$

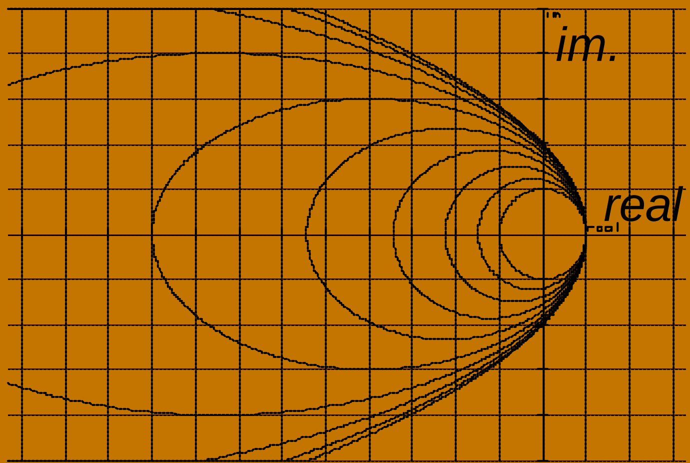


Stability analysis

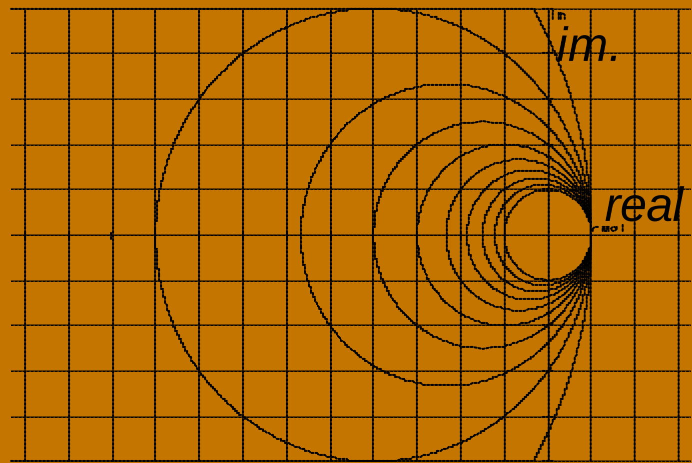
Traffic assignment

γ_k one of the (n) eigenvalues of

$\mathbf{G} = \mathrm{Jac}[\boldsymbol{f}(\mathbf{c}^*)] \times \mathrm{Jac}[\boldsymbol{c}(\mathbf{f}^*)]$



$\alpha = \beta$



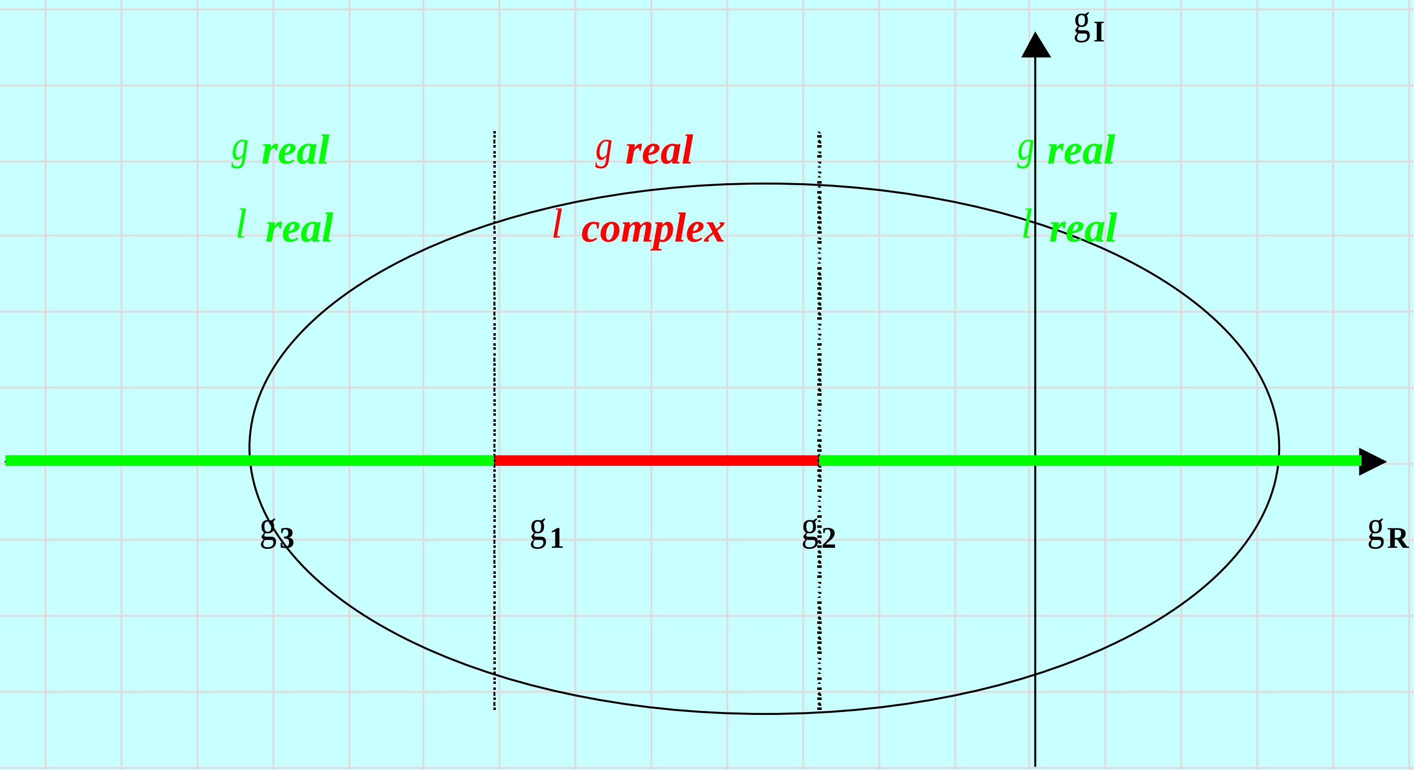
$\beta = 1$

Stability analysis

Traffic assignment

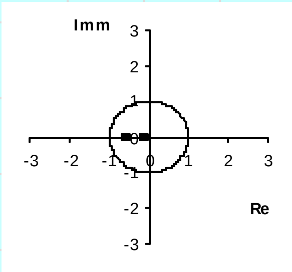
γ_k complex cojugate → λ_j complex conjugate

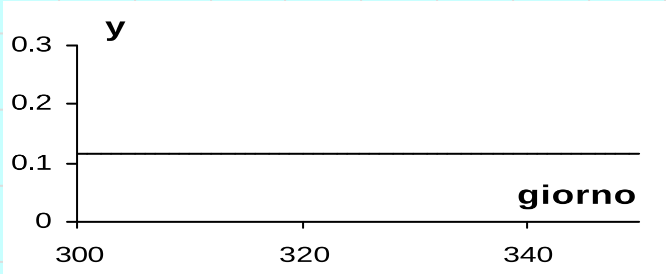
γ_k reali → λ_j real or complex and coniugate

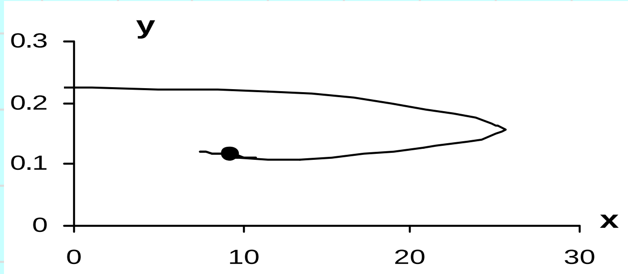


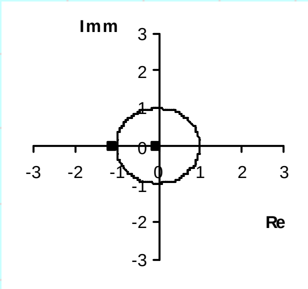
Stability analisys

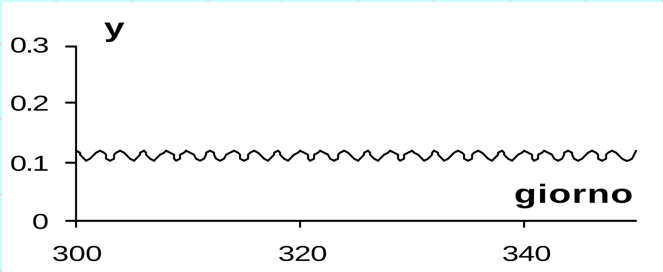
Example: two-link network

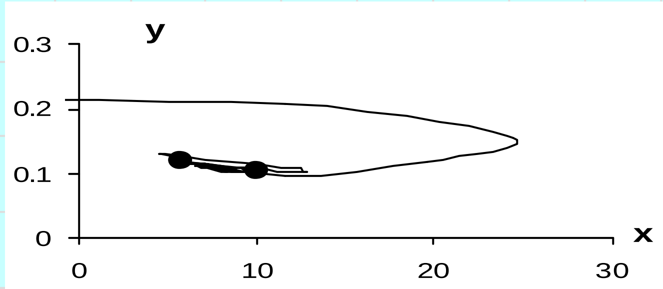


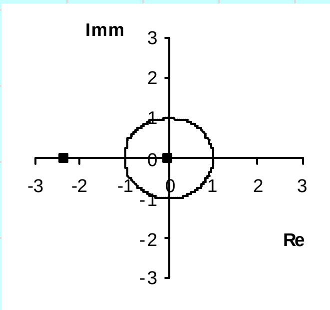


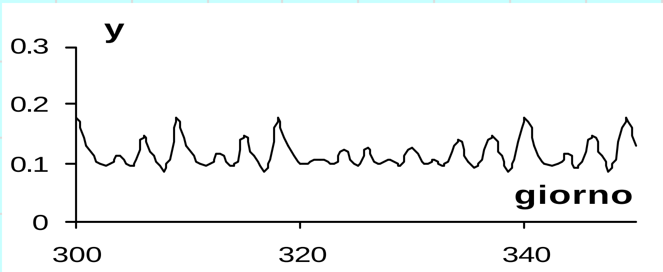


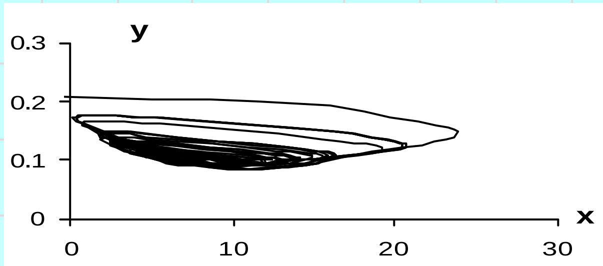








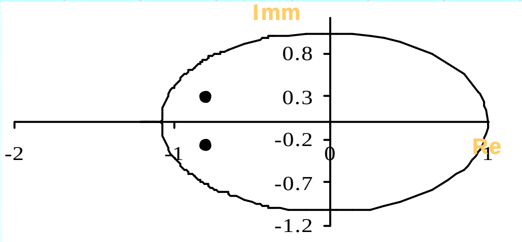


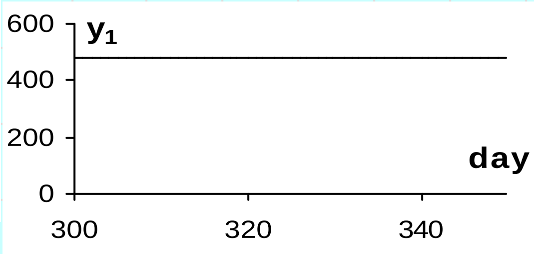


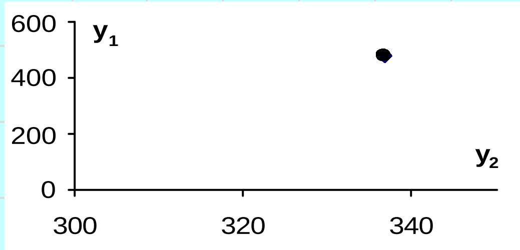
Stability analisys

Example: three-link network

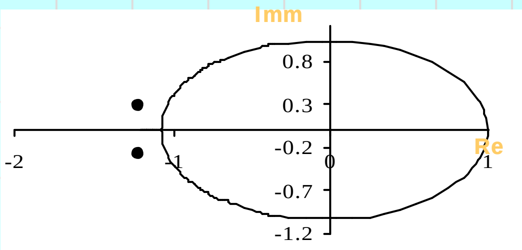
i) θ=0.010 fixed-point attractor

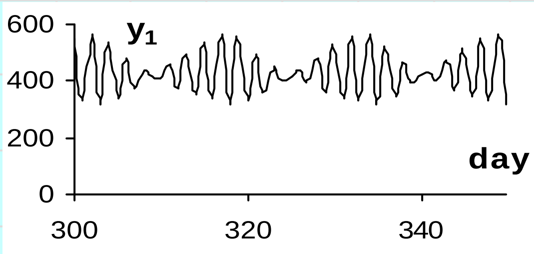


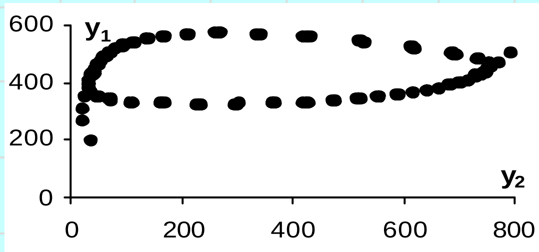




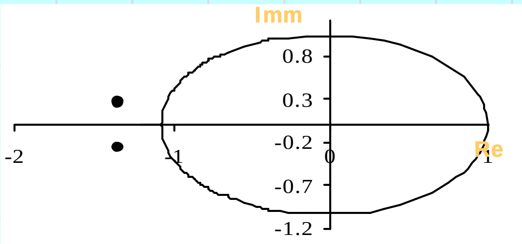
ii) θ=0.015 quasi-periodic attractor

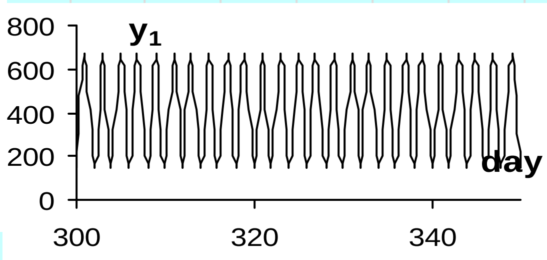


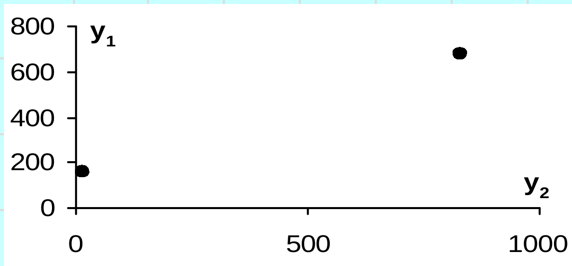




ii) θ=0.018 2-periodic attractor

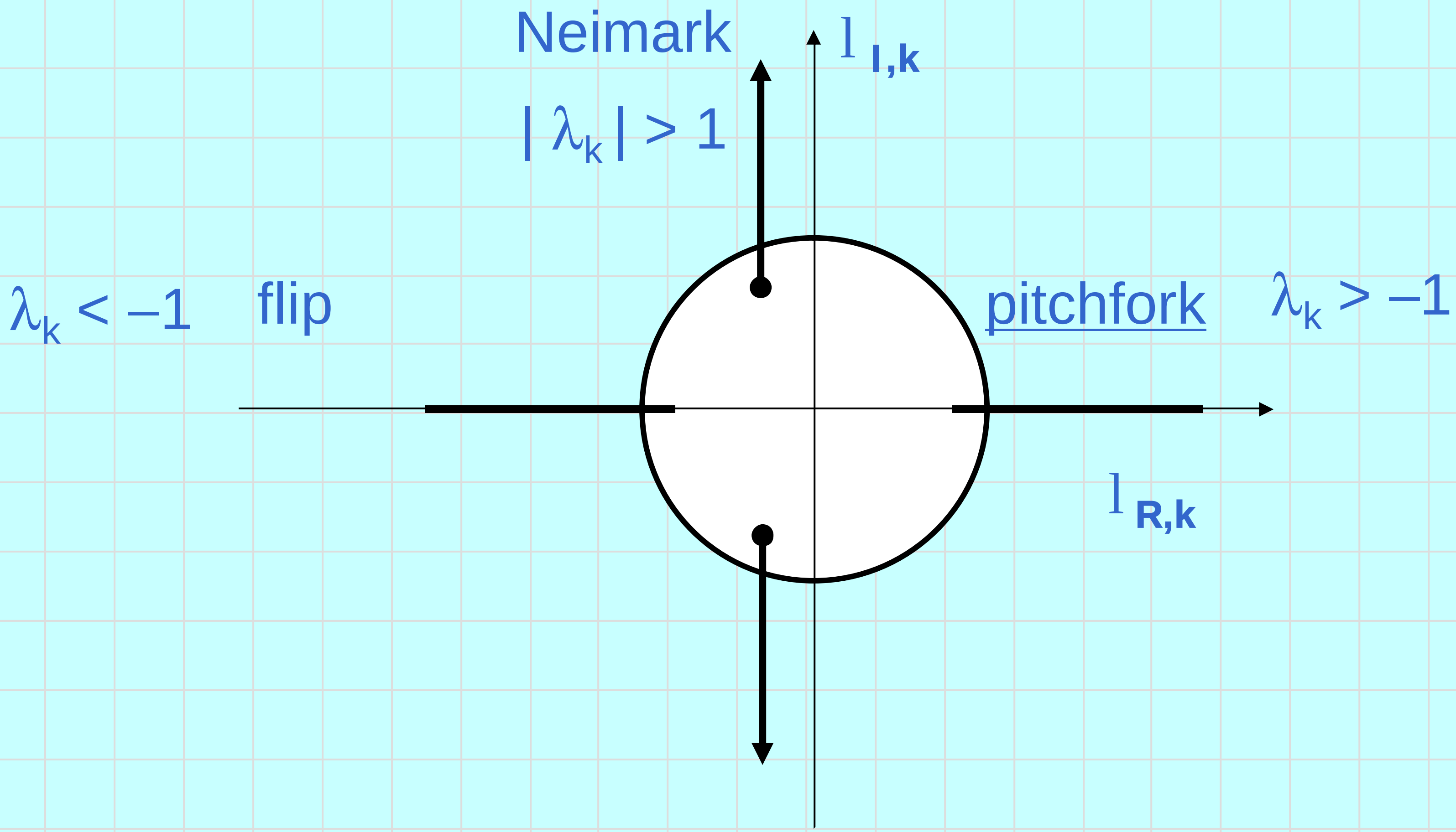






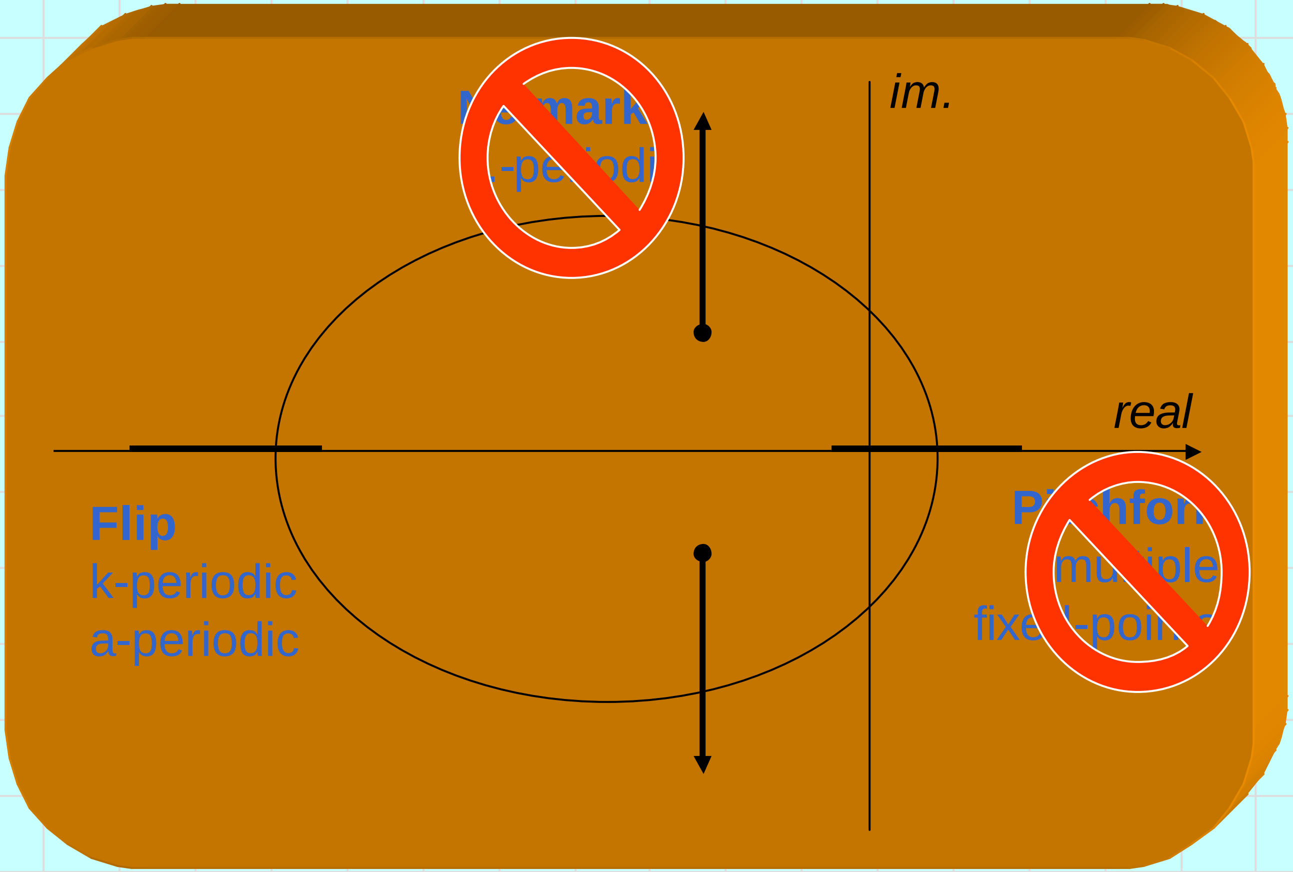
Bifurcation analysis

Mathematical background



Bifurcation analysis

Traffic assignment

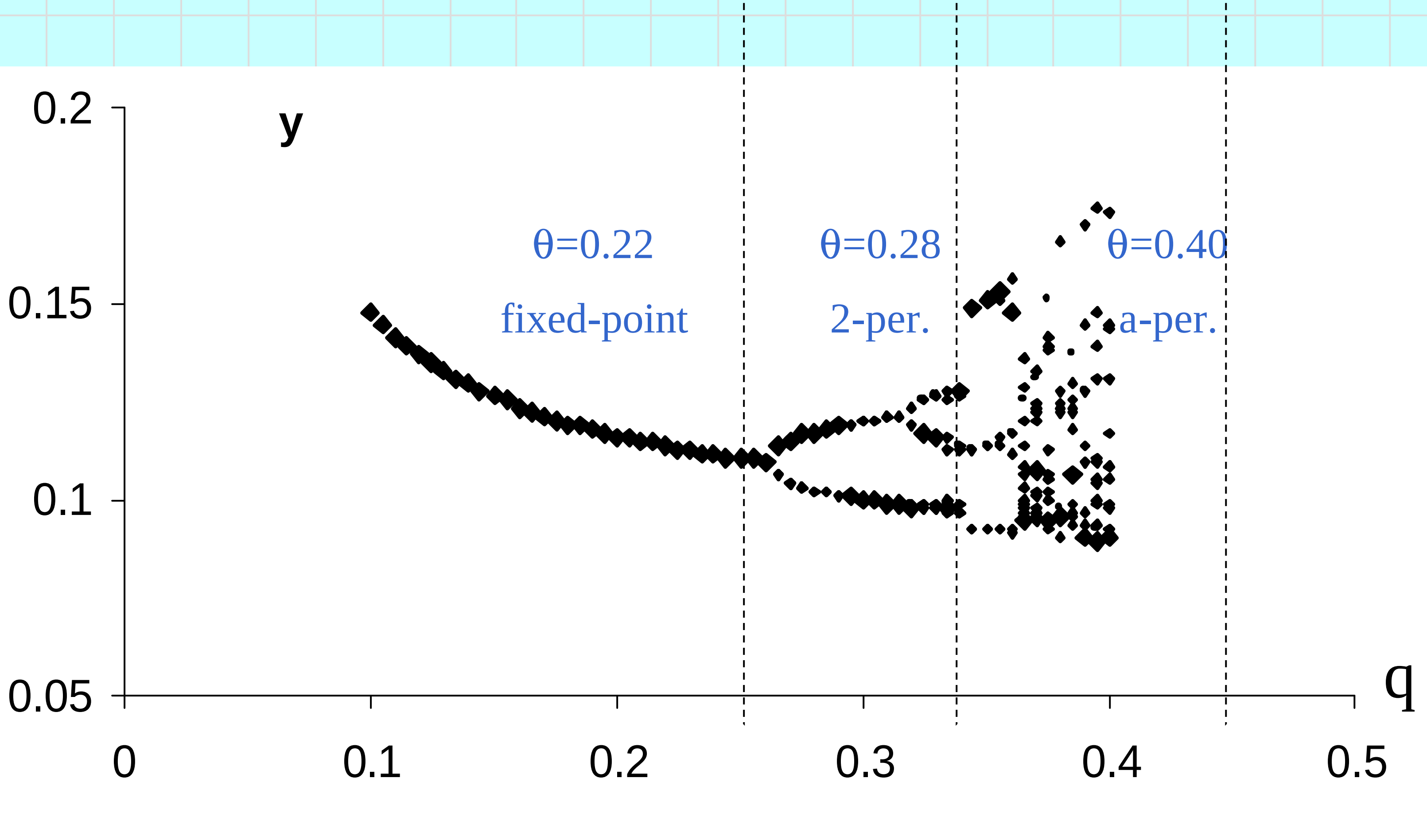


Jac[***f***(**c***)]
sym. neg.
semidef.

Jac[***c***(*)]
sym. pos.
def.

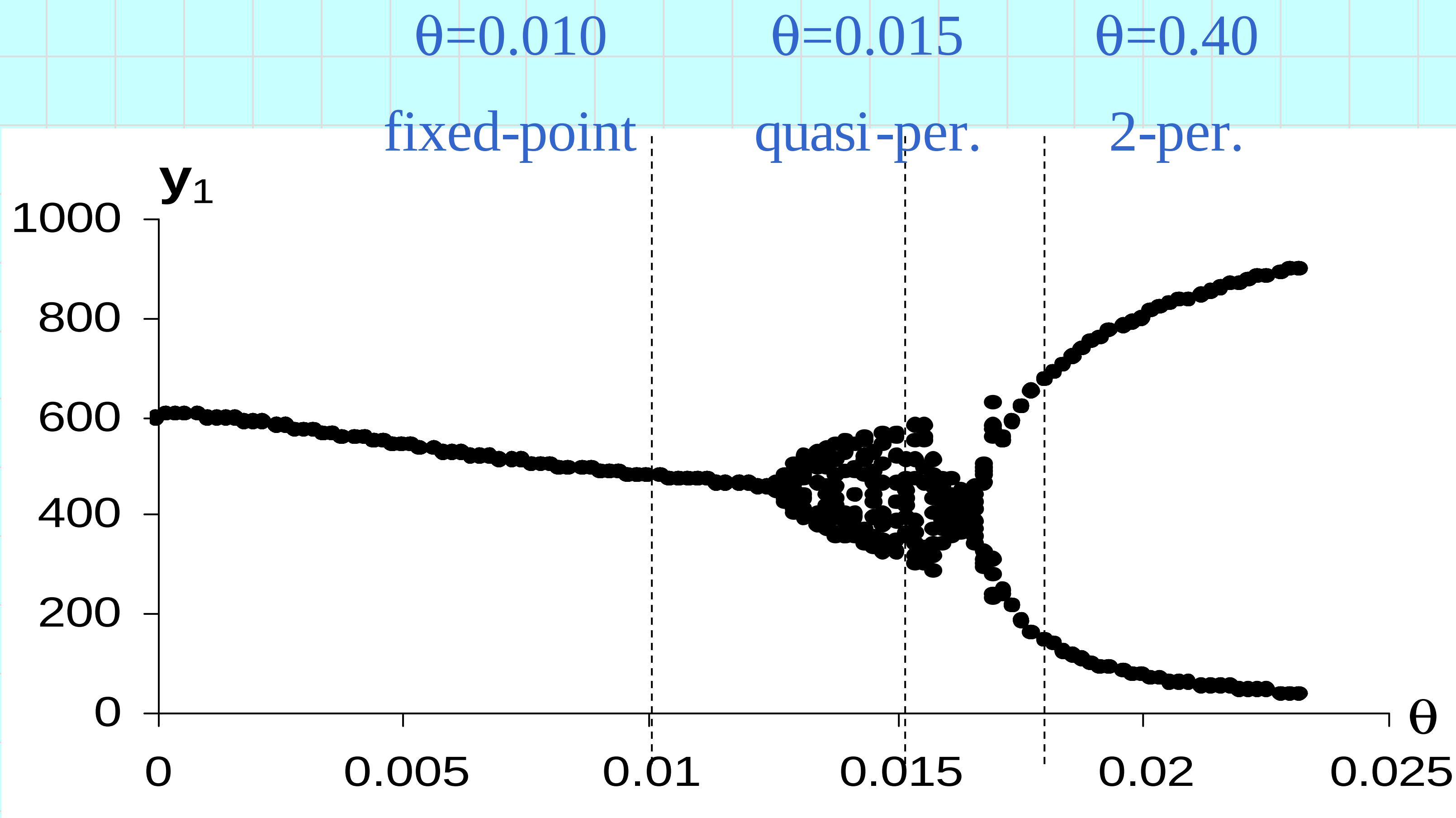
Bifurcation analysis

Example: two link network



Bifurcation analysis

Example: three link network



Attractor identification

Mathematical background

Lyapunov's multipliers

$$\mu_j = \lim_{t\to\infty}(|\lambda^t_j|)^{1/t}$$

λ^t_j j-th eigenvalue of matrix $\mathbf{Jac}[\varphi^t(\mathbf{x}_0)]$

Type	Condition on multipliers
fixed-point	$0 \le \mu_n \le \ldots \le \mu_1 < 1$
k-periodic	$0 \le \mu_n \le \ldots \le \mu_1 < 1$
quasi-periodic	$0 \le \mu_n \le \ldots \le \mu_{k+1} < \mu_k = \ldots.. = \mu_1 = 1$ (the number of $\mu_k = 1$ gives the dimension of the torus)
a-periodic	$0 \le \mu_n \le \ldots \le \mu_1$ $\mu_1\,\mu_2\ldots\ldots\,\mu_n < 1$

Attractor identification

Example: two link network

θ	μ_1	μ_2	# of states
0.22	**0.000000**	**0.000001**	**1**
0.24	0.795186	0.000000	1
0.26	0.000000	0.985271	2
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1	0.836746	0.000000	6

Attractor identification

Example: three link network

θ	μ_1	μ_1	μ_3	μ_4	# of states
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0.010	0.886771	0.000001	0.901406	0.901406	1
0.012	0.955696	0.00000	0.972199	0.972199	1
0.013	0.000000	0.977098	1.000000	1.000000	quasi-periodic
0.014	0.000000	0.965666	1.000000	1.000000	quasi-periodic
0.015	1.000000	1.000000	0.000001	0.961409	quasi-periodic
0.018	0.863079	0.861943	0.000001	0.862810	2

End of presentation

conclusions

stability of equilibrium
cannot be given for granted
equilibrium approach may fail to describe the state of the system relevant to analysis and design

- theoretical results
- examples for simple networks

- results of applications for real-size networks
  - more formally analysis useful
  - calibration methods
  - algorithms for large-scale networks

End of presentation

Research perspectives

- analysis of the case of multiple fixed-point states through results from catastrophe theory
- length of transients before convergence
- fractal measure for a-periodic attractors
- global stability
(numerical results seem to support the conjecture that a local stable fixed-point state is also globally stable)
- more general models
- relationship with stochastic process models

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Fuzzy vs. Random Utility Models: mode choice behaviour analysis

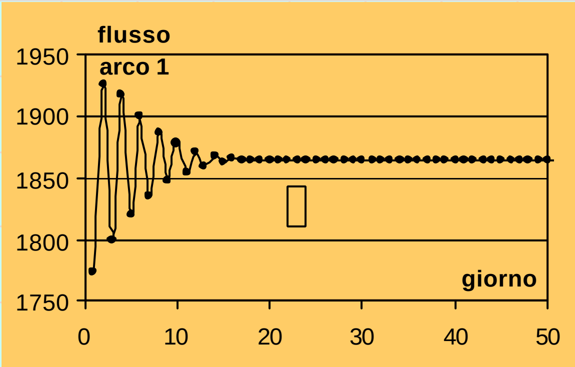
Questions?

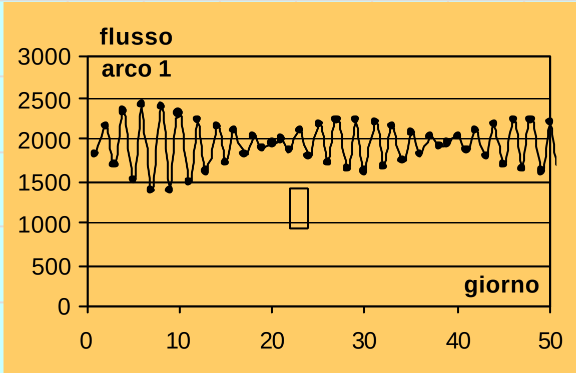
Comments?

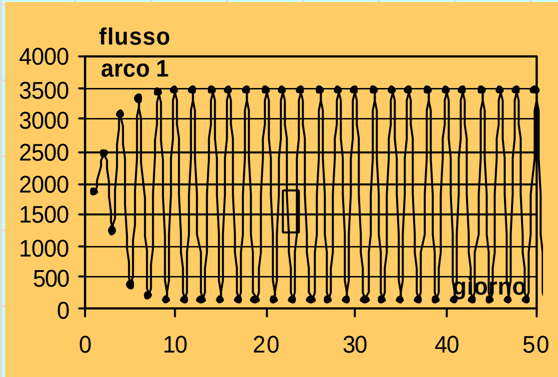
examples

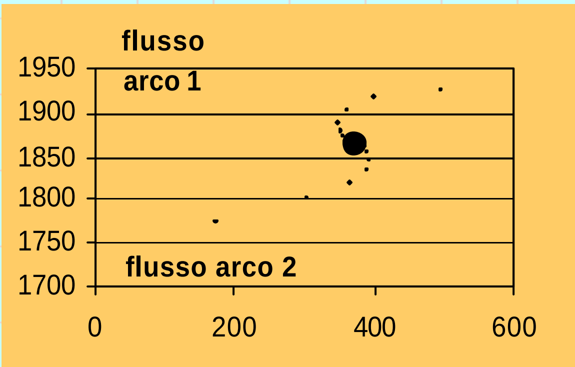
Teta

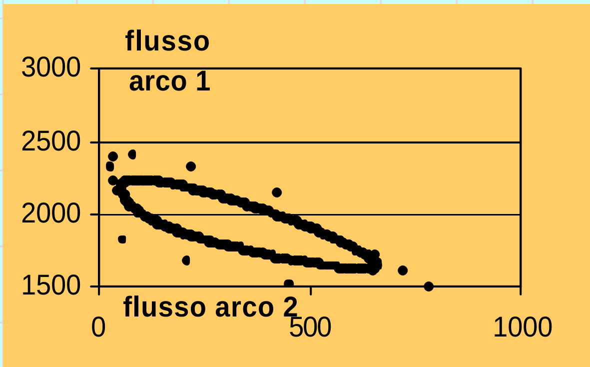
0.05 | 0.077 | 0.11

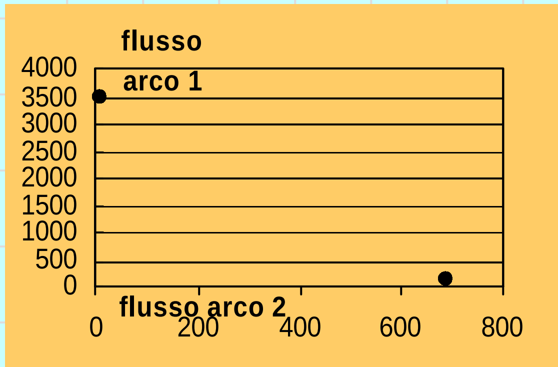








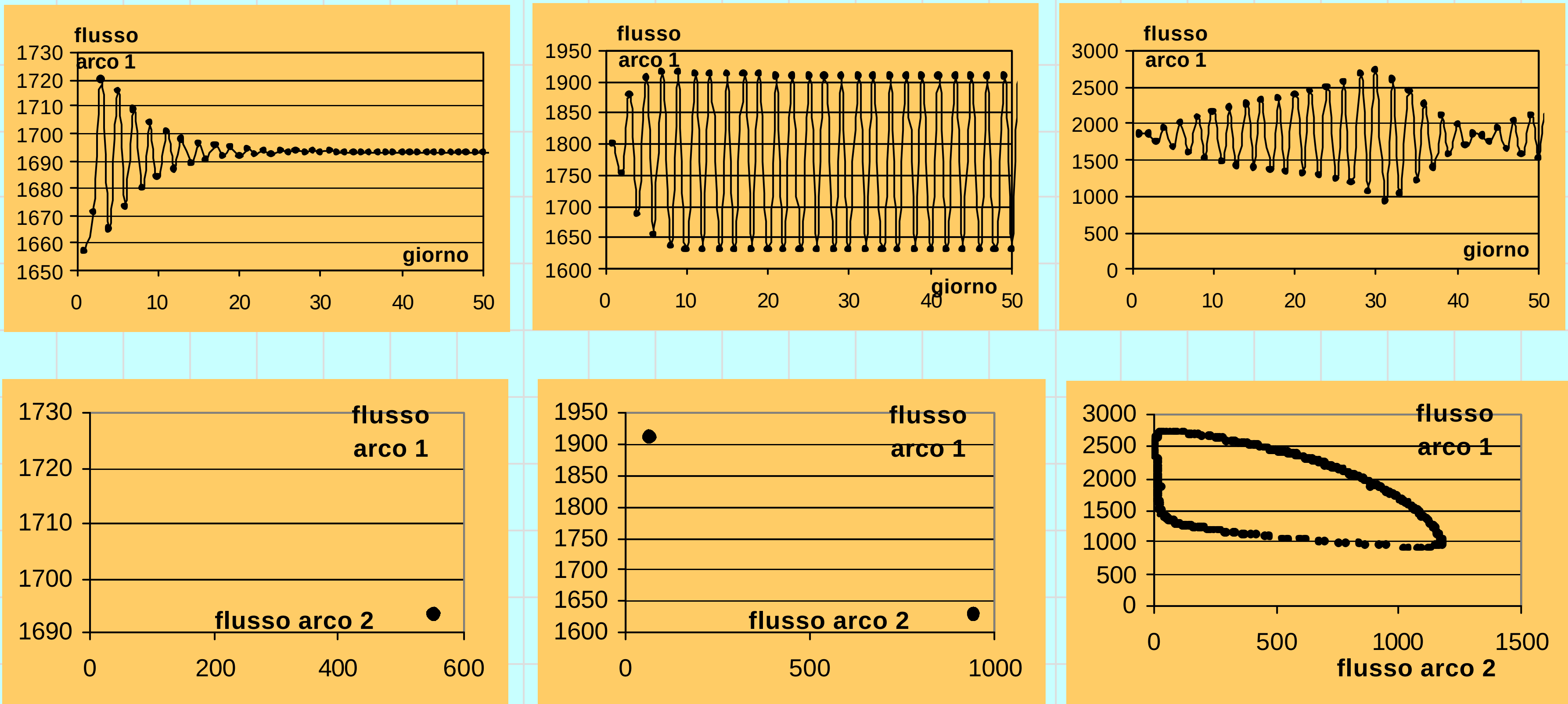




examples

Teta

0.025 | 0.050 | 0.077



examples

Teta

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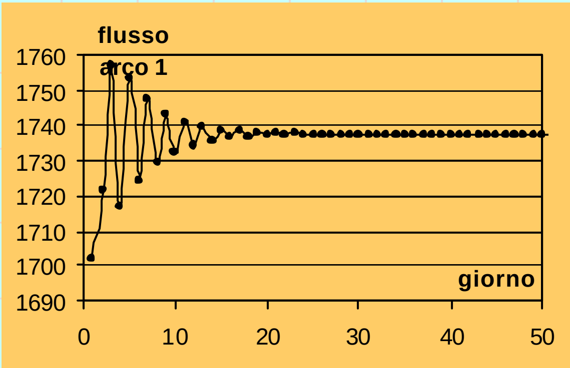

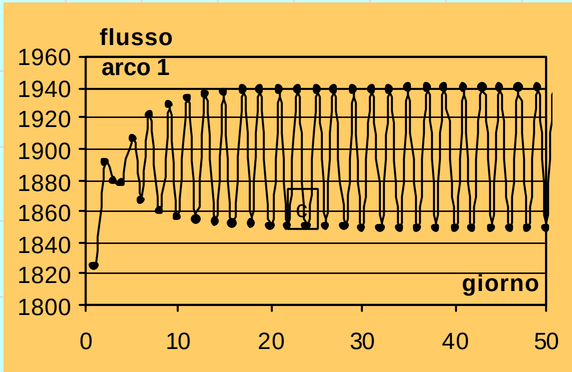

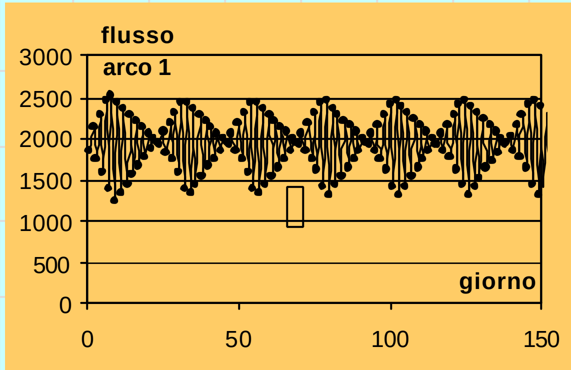

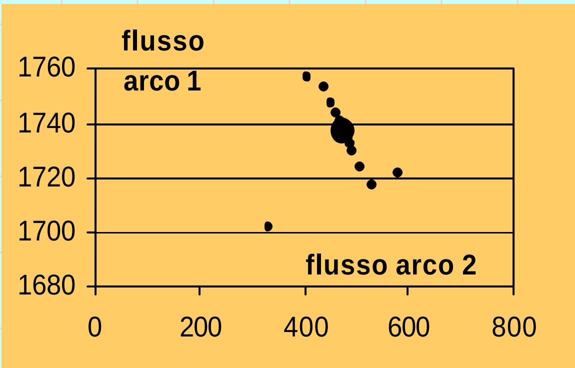

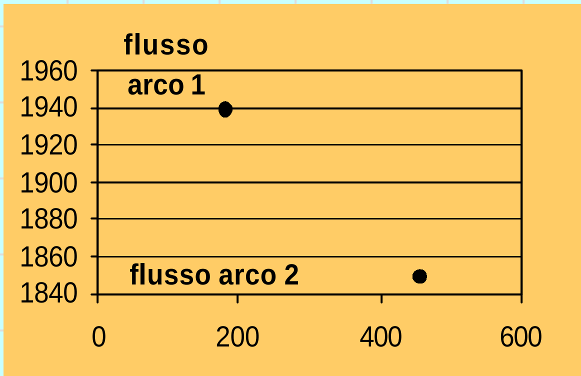

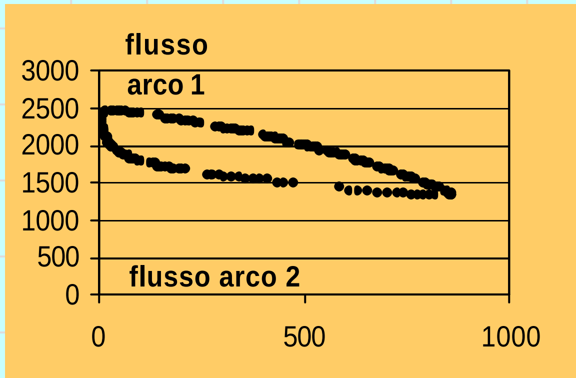

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